# Q1 Standing Waves

- ▽ Classical Waves
  - Standing Waves
  - The Wave Nature of Light
- ▷ Particles or Waves?
- ▷ Basic Quantum Physics
- ▷ The Schrödinger Equation
- ▷ Nuclear Physics

## Chapter Overview

### Section Q1.1: Introduction to the Unit

This unit is focused on *quantum mechanics*, the revolutionary theory of microscopic systems that lies at the foundation of most of 20th-century physics. This theory grew out of the observation that in certain circumstances, *matter behaves like waves*. Each of the unit subdivisions shown in the menu to the left explores a crucial aspect of this great idea. See the section for a more detailed description of each of the five subunits.

### Section Q1.2: Tension and Sound Waves

In this chapter, we will focus primarily on one-dimensional waves that we can describe with a disturbance function $f(x,t)$ of position and time alone. **Tension waves** on a stretched string and **sound waves** in a tube are common and accessible examples of one-dimensional classical waves. A sound wave involves disturbances of the pressure and density of a gas away from the ambient atmospheric pressure and density.

### Section Q1.3: The Superposition Principle

The **superposition principle** for waves states that

> If two traveling waves are moving through a given medium, the disturbance function $f(x,t)$ for the combined wave at any time $t$ and any position $x$ is simply the algebraic sum of the functions $f_1(x,t)$ and $f_2(x,t)$ that describe the individual waves: $f(x,t) = f_1(x,t) + f_2(x,t)$.

This is not always strictly true, but for almost all small-amplitude mechanical waves, it is an excellent approximation.

### Section Q1.4: Reflection

When a medium's characteristics change significantly and suddenly at a certain **boundary,** waves will at least be partially reflected by that boundary. Waves are *completely* reflected at boundaries where their disturbance values are either *fixed* (for example, a string whose end is attached to a fixed point) or *free* (for example, a string whose end is allowed to move freely up and down). The wave reflected from a fixed boundary is *inverted*, but the wave reflected from a free end is *upright*. For sound waves in a tube, an opening in the tube acts as a fixed end on a string (because the air pressure at the opening is constrained to be the same as atmospheric pressure), while a closed end acts as the free end of a string does.

### Section Q1.5: Standing Waves

Sinusoidal waves reflected from a boundary will interfere with incoming waves in such a way as to create a standing wave described by the disturbance function

$$f(x,t) = 2A \sin kx \cos \omega t \tag{Q1.9}$$

Such a wave does not move, but amounts to a fixed sinusoidal disturbance $\sin kx$ whose overall amplitude oscillates with time. The disturbance is always zero at points where $\sin kx = 0$; we call such points **nodes** of the standing wave. The disturbance oscillates maximally at positions where $\sin kx = \pm 1$; we call such points **antinodes** of the standing wave.

When a standing wave is trapped between two fixed boundaries a distance $L$ apart, only waves having certain frequencies will match the specified boundary conditions. For example, standing waves on a string with two fixed ends must go to zero at both boundaries, so $L$ must correspond to an integer number of half wavelengths, and the constraint on the wavelength constrains the standing wave's frequency as well. In general, the frequencies are such that

$$f = \frac{v}{2L}n \qquad \text{where } n = 1, 2, 3, \ldots \qquad \left(\begin{array}{c}\text{when the boundaries are analogous}\\ \text{to either two free or two fixed ends}\end{array}\right) \tag{Q1.12a}$$

$$f = \frac{v}{4L}n \qquad \text{where } n = 1, 3, 5, \ldots \qquad \left(\begin{array}{c}\text{when the boundaries are analogous}\\ \text{to one fixed and one free end}\end{array}\right) \tag{Q1.12b}$$

**Purpose:** These equations describe the frequencies $f$ of the fundamental modes of standing waves in a medium between two reflecting boundaries.

**Symbols:** $L$ is the distance between the boundaries, $v$ is the speed of traveling waves in the medium, and $n$ is a positive integer (a positive *odd* integer in the second case).

**Limitations:** This equation applies only when the medium is uniform and the waves are essentially one-dimensional. The boundaries must be perfectly reflecting if the wave is to sustain itself.

We call the allowed standing waves in such a system the system's **normal modes** (of oscillation). We call the $n = 1$ normal mode the system's **fundamental mode** and its frequency the system's **fundamental frequency.** We call modes where $n > 1$ the **harmonics** of the fundamental mode.

## Section Q1.6: The Fourier Theorem

The Fourier theorem implies that we can think of *any* disturbance as being a sum of sinusoidal disturbances, and in particular, a disturbance in a medium between reflecting boundaries can be written as a superposition of waves corresponding to the system's normal modes. The section illustrates this in the case of a square wave (whose disturbance value is $+A$ for one-half of the distance between the boundaries and $-A$ for the other half).

## Section Q1.7: Resonance

A disturbance of a medium between two boundaries most effectively transfers energy to that system if the frequency of the disturbance most closely matches one of the system's normal-mode frequencies. For example, a violin bow sliding on a string disturbs the string in complicated random ways, which the Fourier theorem teaches us we can think of as being a superposition of sinusoidal waves with many frequencies. Those frequencies that match the string's normal-mode frequencies will transfer energy to the string, causing the string to vibrate in a mixture of normal-mode frequencies that we hear as a fundamental tone "colored" by higher harmonics.

## Q1.1 Introduction to the Unit

Quantum mechanics is based on the idea that matter behaves like waves

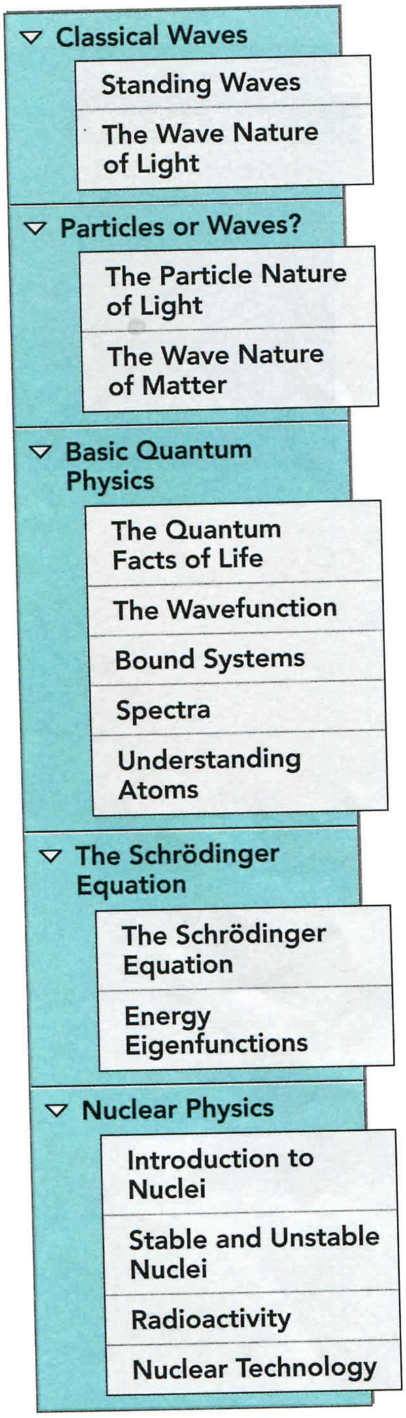



**Figure Q1.1**
The structure of unit Q.

Perhaps the most important revolution in 20th-century physics resulted from the discovery that newtonian mechanics was unable to adequately describe the behavior of systems of particles roughly the size of atoms and smaller. The behavior of such systems is better described by the theory of *quantum mechanics,* which first began to take its modern shape in the late 1920s after decades of work by many physicists. Quantum mechanics is the foundation on which virtually all modern physics and chemistry rest.

Quantum mechanics is based on the idea that under certain conditions, objects (even those considered to be point particles, such as electrons) exhibit *wavelike* behavior: this strange thought is the "great idea" of this unit. The wavelike aspect of matter has a variety of surprising implications. For example, one implication is that the energy of a bound system of particles (such as an atom) must be *quantized* (that is, its energy can have only certain discrete and distinct values). This provides the key for understanding atomic spectra and certain details of molecular, atomic, and nuclear structure that have no newtonian explanation.

The world seen through the eyes of quantum mechanics is a very strange one, where a "particle" seems to follow more than one path in getting from point $A$ to point $B$ and interferes with itself, where the best predictions we can make are statistical, and where the simple act of observing a system irrevocably changes its physical state. Indeed, quantum mechanics is so strange that almost no one thinks that physicists completely understand it yet. Even so, it has proved to be indispensable in modern physics.

The goals of this unit are to provide a limited introduction to the theory of quantum mechanics, specifically tracing how the wavelike aspect of matter is linked to the phenomenon of energy quantization and its consequences for the structure of the atomic nucleus. The unit has five major subdivisions, as shown in Figure Q1.1:

1. *Classical waves* (chapters Q1 and Q2). The theory of quantum mechanics is based on making an analogy between the wavelike aspects of matter and "classical" waves that we can more directly view in nature (such as water and sound waves). This subdivision provides foundations for understanding the analogy by discussing the behavior of classical waves and the similar behavior of electromagnetic (light) waves.
2. *Particles or waves* (chapters Q3 and Q4). Once we have come to understand how classical waves behave, we can explore more fully how both light and matter behave in ways that are both wavelike and particlelike. This subdivision lays the foundation for a *quantum* theory that embraces the wavelike nature of matter.
3. *Basic quantum physics* (chapters Q5 through Q9). In this subdivision, we will explore the basic structure of quantum mechanics, focusing on why the wavelike behavior of matter implies that the energy of a bound system must be quantized and exploring the implications of this idea for atomic and molecular spectra.

These subdivisions comprise the unit's core. The remaining subdivisions are independent of each other and explore extensions and applications of this core.

4. *The Schrödinger equation* (chapters Q10 and Q11) extends the ideas presented in the core chapters to create a more sophisticated model of the quantum behavior of bound systems. This subunit closes with a discussion of the phenomena of *tunneling* and the *covalent bond.*

5. *Nuclear physics* (chapters Q12 through Q15). This subunit focuses on how the ideas presented in the core chapters illuminate the structure and behavior of atomic nuclei. Studying nuclei will also give us a chance to review and apply some results from units R and E and an opportunity to discuss radioactivity and nuclear power, which have important technological applications and societal implications.

## Q1.2 Tension and Sound Waves

Review of basic wave concepts

A **classical wave,** you will recall from unit E, is fundamentally a *disturbance that moves through a medium* that remains basically at rest. The most common examples of classical waves are *water waves* that disturb the surface of a body of water, *tension waves* that move along a stretched string or spring, and *sound waves* that move through fluids and/or solid objects.

In this chapter, we will focus our attention on waves that move in one dimension (which we take to be the $x$ axis) and that we can accurately describe by some function $f(x, t)$ that expresses the disturbance caused by the wave as a function of position $x$ along the $x$ axis and time $t$ (note that this function essentially ignores the shape of the wave in the $y$ and $z$ directions).

A **tension wave** on a stretched string is an excellent example of such a one-dimensional wave. In this case, the function $f(x, t)$ describes the transverse displacement from its equilibrium position (which we assume to lie along the $x$ axis) at position $x$ and time $t$ (see figure Q1.2a). Such a wave can clearly travel only along the string (and thus the $\pm x$ direction), so a simple function of $x$ and $t$ completely describes such a wave.

Sound waves

A **sound wave** in the air inside a narrow cylindrical tube is another example of a one-dimensional wave. A sound wave is a fluctuation in the density of the medium, so in this case, the function $f(x, t)$ describes the density of the air above or below its equilibrium value (see figure Q1.2b).† Since

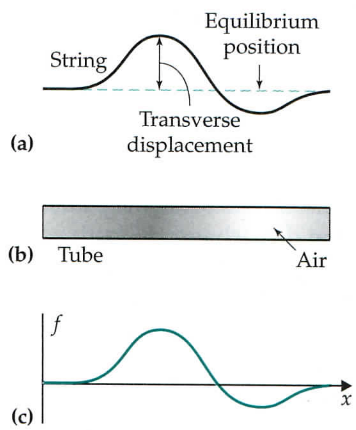



**Figure Q1.2**
(a) A transverse wave on a string. (b) A sound wave in a tube. (c) The mathematical function describing either of the two waves.

†Alternatively, one can describe a sound wave in terms of the extent to which air molecules are displaced from their rest position by the passing wave. While this displacement representation has a few advantages, I think that the density/pressure representation is more intuitive in general. Rather than confuse the issues by discussing both representations, I will use the density/pressure representation exclusively.

the *pressure* of a gas depends strongly on its density, we can also think of a sound wave in a tube as causing a variation in the air pressure in the tube. If the tube is fairly narrow, the variation in the density across the tube's width is negligible, and the wave is adequately described by a function of $x$ and $t$ alone.

## Q1.3 The Superposition Principle

The thing that most sharply distinguishes how waves behave from how particles behave (and thus the aspect of wave behavior that is most crucial in this unit) is how passing waves *interfere* with each other as they overlap. Our fundamental model for understanding this interference is the **superposition principle** for waves, which states that

Statement of the superposition principle

> If two traveling waves are moving through a given medium, the function $f(x,t)$ that describes the combined wave at any time $t$ and any position $x$ is simply the algebraic sum of the functions $f_1(x,t)$ and $f_2(x,t)$ that describe the individual waves: $f(x,t) = f_1(x,t) + f_2(x,t)$.

The superposition principle for waves is a theoretical *assertion* about how interfering waves behave. We find experimentally, however, that *most* of the waves encountered in nature obey this principle. Waves that do obey this principle are called **linear** waves, and those that don't are called **nonlinear** waves. In a given medium, waves that represent *small* disturbances typically obey the superposition principle. For example, the gentle sound waves generated by conversation (or even loud music) obey the superposition principle to a high degree of accuracy, but shock waves (produced by an explosion or a jet moving at supersonic speeds) noticeably do not. In this text, we will study only linear waves.

The superposition of two traveling waves is illustrated in figure Q1.3. In this diagram, a traveling pulse wave moving in the $+x$ direction meets a weaker pulse wave traveling in the opposite direction. The figure shows the pulses' motions using a series of "snapshots" arranged vertically, like a spacetime diagram, with time increasing *upward*. Each graph's vertical axis represents the degree to which the medium is disturbed from its normal value. In the case of a tension wave on a string, this axis would correspond to the string's transverse displacement; in the case of a sound wave, it would correspond to the air pressure in the wave compared to normal air pressure, and so on. Figure Q1.3a shows the case where the disturbance in both cases is positive. Figure Q1.3b shows the case where one of the disturbances is negative (that is, in the wave, the quantity represented by the vertical axis is less than the normal value of that quantity in the medium). In each case and at all times, we find the function representing the combined wave by simply adding the functions representing the two pulses.

This implies that waves travel through each other

An important implication of the superposition principle, as illustrated in figure Q1.3, is that linear traveling waves can pass through each other without modification. For example, two pebbles dropped in a pond produce rings of ripples that pass through each other without affecting each other. Similarly, two widely separated people calling to each other can hear and interpret the details of each other's calls even though the two sets of sound waves may cross in transit.

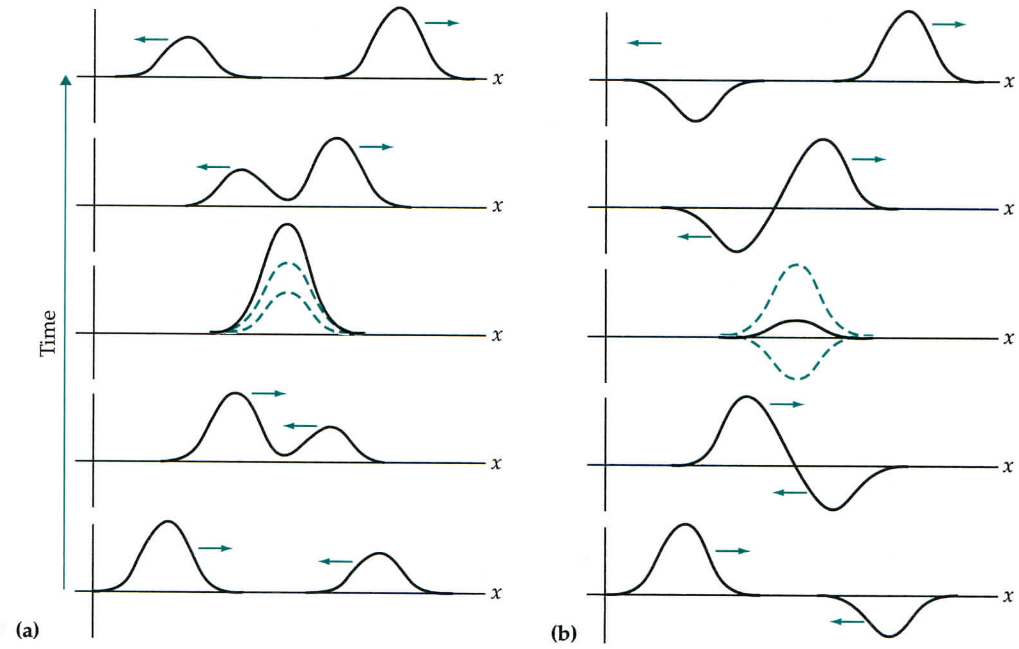
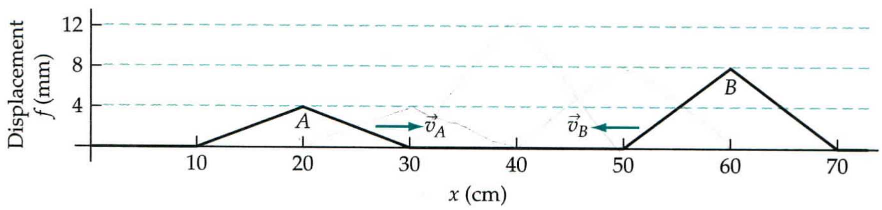



**Figure Q1.3**
Illustration of the superposition principle for traveling pulse waves. Read this diagram from the bottom up. Note that in (b), where the waves have opposite signs, the disturbance when the waves cross is smaller than the absolute magnitude of either wave.

## Exercise Q1X.1

Two transverse wave pulses, $A$ and $B$ on the diagram below, are moving with the same constant speed of 10 cm/s but in opposite directions along a stretched string. The graph shows the shape of the function describing the two waves at $t = 0$. On the same graph, sketch the function describing the combined wave at $t = 2$ s and $t = 3$ s.




## Q1.4 Reflection

Extreme cases of boundaries between media

When a traveling wave encounters a *boundary*, part of or all the wave will be reflected at that boundary. A **boundary** in this context is somewhere where the medium's characteristics suddenly change. A wave on a string encounters a boundary if the mass density of the string changes at a point. A sound wave moving through a tube encounters a boundary if the tube suddenly widens or narrows. A light wave moving through transparent glass is reflected at both the front and back surfaces of the glass, because glass has different characteristics than air as a medium for light waves.

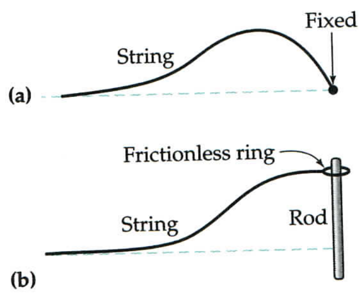



**Figure Q1.4**
A pulse wave encountering (a) a fixed end and (b) a free end. (In the latter case, the ring and rod ensure that the string's displacement at the free end is entirely transverse.)

We can understand this phenomenon better if we consider two extreme cases. Imagine a traveling wave on a stretched string, and let one end of the string be *fixed*, so that it cannot displace at all in response to the wave. (You can think of this situation as being the extreme case of one string being connected to a second string of infinite mass density.) The wave cannot move beyond the fixed point, but its energy must go somewhere. The only way that the wave's energy can be conserved is if the boundary entirely reflects the wave.

The opposite extreme is seen when one end of the string is completely *free* (this is the extreme case of one string being connected to a second string with zero mass density). Since again the wave cannot move at all beyond this boundary, its energy will be conserved only if the boundary entirely reflects it.

Figure Q1.4 illustrates fixed and free boundaries for a string.

Even though the wave is completely reflected at both the fixed and free boundaries, the wave reflected from a fixed end is *opposite* in sign to that of the initial wave, whereas the wave reflected from a free end has the *same* sign (see figure Q1.5). Why?

### A model for understanding reflection

We can use the following model to understand this. Pretend that the string does *not* end at the boundary point $x_B$, but rather continues past that point. Let us imagine that as our original pulse wave moves toward $x_B$, we create another pulse wave the same distance from $x_B$ on the other side whose shape is the mirror image of the first wave, whose sign is opposite, and which moves toward $x_B$ at the same speed. Since these waves always have equal magnitudes but opposite signs at $x_B$, they will exactly cancel each other there, keeping the string at $x_B$ fixed at all times as they move past each other.

Now the string to the *left* of $x_B$ cannot tell whether the string is motionless at $x_B$ because it is fixed there or because another upside-down mirror-image wave happens to be coming in from the right. Therefore, it must behave in the same way in either case. Since in our imaginary case the upside-down mirror image wave will continue to move toward the left, this must be what happens in the fixed-end situation also. Therefore, we see that the fixed end must reflect an upside-down, mirror-image version of the wave that hits it (see figure Q1.5a).

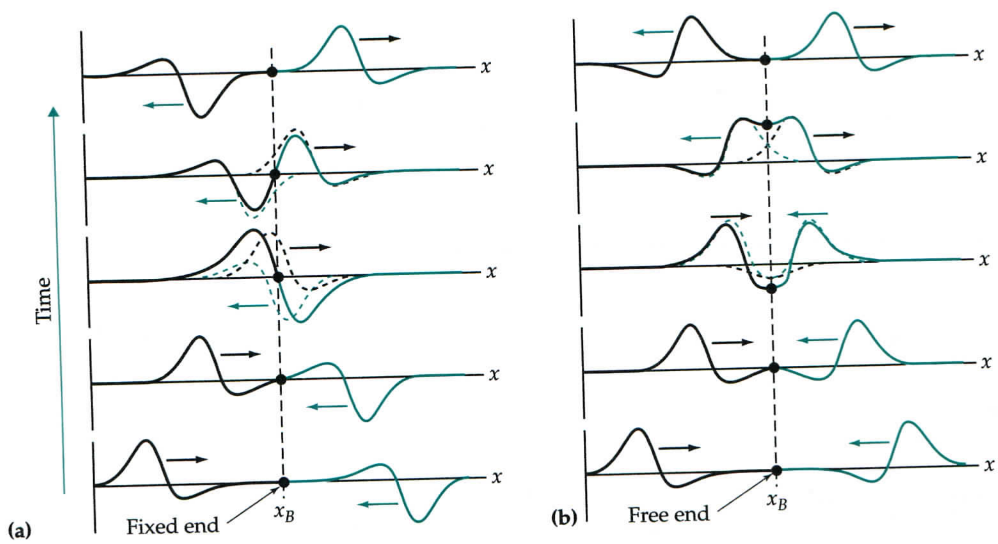



**Figure Q1.5**
A model for understanding reflection of pulse waves on a string. We *imagine* the string to continue beyond the boundary (the imaginary part is in color here) and that the pulse is met at the boundary by a mirror-image pulse. Dotted lines show the individual pulse waves; solid lines show their sum. Read the diagram from the bottom up.

We can use the same model to help us understand reflection from a free end. Here, the trick is to understand that the string's *slope* as we approach the free end must go to zero. Why? Let the tension on the string be $F_T$ and its mass per unit length be some constant $\mu$. Consider a tiny hunk of string of length $dx$ at the end of the string: the mass of this hunk will be $m = \mu\, dx$. The transverse (vertical) component of the force exerted by the rest of the string on this part of the string will be $-F_T \sin\theta$, which is approximately $-F_T \tan\theta = F_T \cdot$ (slope of the string) if $\theta$ is small (see figure Q1.6). As $dx \to 0$, $m \to 0$, so this transverse force must also go to zero so that the hunk's acceleration does not grow to infinity in this limit. Thus, the *slope* at the end of the string must go to zero as $dx$ gets small.

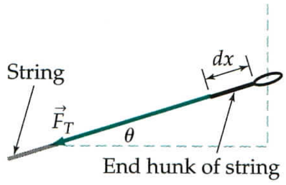



**Figure Q1.6**
Close-up of the free end of a string.

Now pretend that the string continues past the boundary, and imagine that as our original pulse moves toward the boundary point, a mirror-image *upright* pulse also moves toward the boundary point from the other side. Since the slopes of the original pulse and its mirror-image are always equal in magnitude and opposite in sign at $x_B$, the combined wave's slope will always add to zero at $x_B$, just as if the end were free. Since the left part of the string cannot tell whether the end is really free or just *behaving* as if it were free because of a pulse coming in from the right, it will behave the same way in either case. Thus the free end must reflect an *upright* mirror image of any wave that encounters it (figure Q1.5b).

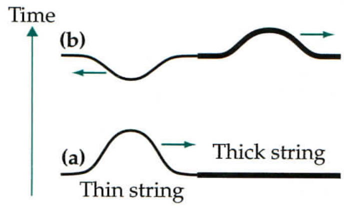



**Figure Q1.7**
When a wave on a string passes a boundary where the string's thickness changes, part of the wave is reflected and part is transmitted. (Read this diagram from the bottom *up*.)

If the boundary is intermediate between these extremes (a connection between a thin and thick string, for example), a wave will be partially transmitted and partially reflected (see figure Q1.7). The reflected part of the wave is upside down if the medium beyond the boundary is more like a fixed point (a denser string, for example), and right side up if it is more like a free end (a less dense string, for example).

The *open* end of a cylindrical tube is to a sound wave what a fixed end is to a wave on a string (contrary to what one might intuitively expect!). This is so because while the air pressure can vary dramatically when the air is trapped in the tube (and thus cannot escape regions of high pressure easily), air under even a small amount of pressure near an open end can just expand into the surrounding atmosphere, dissipating that pressure. Therefore the air pressure is essentially *fixed* at the value of atmospheric pressure at the tube's open end.

On the other hand, the air's density and pressure can vary freely at a closed end of the tube (indeed, the pressure can become quite high as air is crammed against the closed end by a wave). A *closed* end to a tube is thus to a sound wave inside the tube what a *free* end is to a wave on a string. This is illustrated in figure Q1.8. (Make sure that you remember this analogy!)

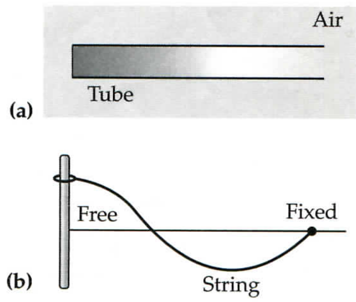



**Figure Q1.8**
The closed and open ends of a tube are to a sound wave inside like the free and fixed ends of a string are to a wave on the string.

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### Exercise Q1X.2

A ripple on the surface of water in a tub will reflect off the solid wall of the tub. Will the reflected wave be an upright or inverted version of the original wave?

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## Q1.5 Standing Waves

Up to this point, we have been considering mostly pulselike traveling waves (because the ideas of superposition and reflection are easier to think about and illustrate in the case of pulse waves). However, many naturally occurring waves are similar to *sinusoidal* waves.

A review of the characteristics of sinusoidal waves

In unit E, we saw that we can describe a one-dimensional sinusoidal traveling wave by the mathematical function

$$f(x,t) = A\sin(kx - \omega t) \tag{Q1.1}$$

where $A$ is the amplitude of the wave and $k$ and $\omega$ (which we call the wave's *wavenumber* and *angular frequency,* respectively) are constants related to the wave's wavelength $\lambda$ and period $T$ by the expressions

$$k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{T} \tag{Q1.2}$$

A wave described by equation Q1.1 is a *traveling* sinusoidal wave. We can see this by focusing on a given crest of the wave, say, the one where the phase value in the parentheses is $\pi/2$. This crest's location at all times is given by

$$\frac{\pi}{2} = kx_{\text{crest}} - \omega t \qquad \Rightarrow \qquad x_{\text{crest}} = \frac{\pi}{2k} + \frac{\omega}{k}t \tag{Q1.3}$$

Taking the time derivative of this equation, we find that the crest's $x$-velocity is

$$v_x = \frac{dx_{\text{crest}}}{dt} = 0 + \frac{\omega}{k} = +\frac{\omega}{k} \tag{Q1.4}$$

So we see that the wave moves in the $+x$ direction with speed $\omega/k$.

Using the same kind of approach, you should be able to show that

$$f(x,t) = A\sin(kx + \omega t) \tag{Q1.5}$$

describes a sinusoidal wave that moves in the $-x$ direction with speed $\omega/k$.

### Exercise Q1X.3

Verify this last claim.

### Exercise Q1X.4

Show that the speed of the wave described by either equation Q1.1 or equation Q1.5 can also be written (where $f \equiv 1/T$ is the wave's frequency).

$$v = \frac{\lambda}{T} = \lambda f \tag{Q1.6}$$

A standing wave

To understand

Now imagine that we have a string whose left end (at $x = 0$) is fixed. Imagine then that we start a sinusoidal wave moving leftward toward this fixed point. When the wave reaches this fixed point, it will reflect off the fixed end, creating an *inverted* sinusoidal wave moving to the right. In the region where the left- and right-going waves overlap, the total wave function (according to the superposition principle) is

$$f(x,t) = A\sin(kx + \omega t) + A\sin(kx - \omega t) \tag{Q1.7}$$

The first term on the right represents the original left-going wave, while the second term represents the inverted reflected wave. The latter does represent an *inverted* wave, even though its sign in the formula is positive: note that at the boundary point $x = 0$, the second term becomes $A\sin(-\omega t) = -A\sin(+\omega t)$, so it exactly cancels the first term at all times, keeping $f(0,t) = 0$ (consistent with the fact that this point is fixed).

Now there is a trigonometric identity

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \text{(Q1.8)}$$

Using this identity, you can show that we can rewrite equation Q1.7 as follows:

$$f(x,t) = 2A \sin kx \cos \omega t \qquad \text{(Q1.9)}$$

## Exercise Q1X.5

Verify equation Q1.9.

Note that this is *not* the equation of a traveling wave: rather this is the equation of a *stationary* sinusoidal wave $f(x) = \sin kx$ multiplied by a factor $2A\cos\omega t$ that oscillates with time (see figure Q1.9). We call such a wave a **standing wave.**

Note that the string's displacement function $f(x,t)$ is zero at *all* times wherever $\sin kx = 0$. We call these positions of zero movement **nodes.** On the other hand, points where $\sin kx = \pm 1$ oscillate up and down with a larger amplitude than any other points on the string: we call these positions **antinodes.**

Standing waves on a string with boundaries at both ends

If both ends of the string are fixed a distance $L$ apart, sinusoidal waves can bounce back and forth between those ends, creating a self-sustaining standing wave (figure Q1.10 displays such a standing wave on an actual vibrating string). The requirement that the string be fixed at *both* ends, however, implies that the standing waves can have only certain special wavelengths: since $\sin kx$ has to be zero at $x = L$ as well as $x = 0$, we must have

$$kL = n\pi \quad \Rightarrow \quad k = \frac{2\pi}{\lambda} = n\frac{\pi}{L} \quad \Rightarrow \quad L = n\frac{\lambda}{2} \qquad \text{(Q1.10)}$$

where $n$ is some (nonzero) integer. As illustrated in figure Q1.11, this essentially means a sinusoidal standing wave on this string has to fit exactly $n$ half-wavelengths between the two fixed endpoints. The frequency of oscillation $f$ of such a standing wave is given by

$$f = \frac{\omega}{2\pi} = \left(\frac{\omega}{k}\right)\frac{k}{2\pi} = \frac{v}{2L}n \qquad \text{(Q1.11}a\text{)}$$

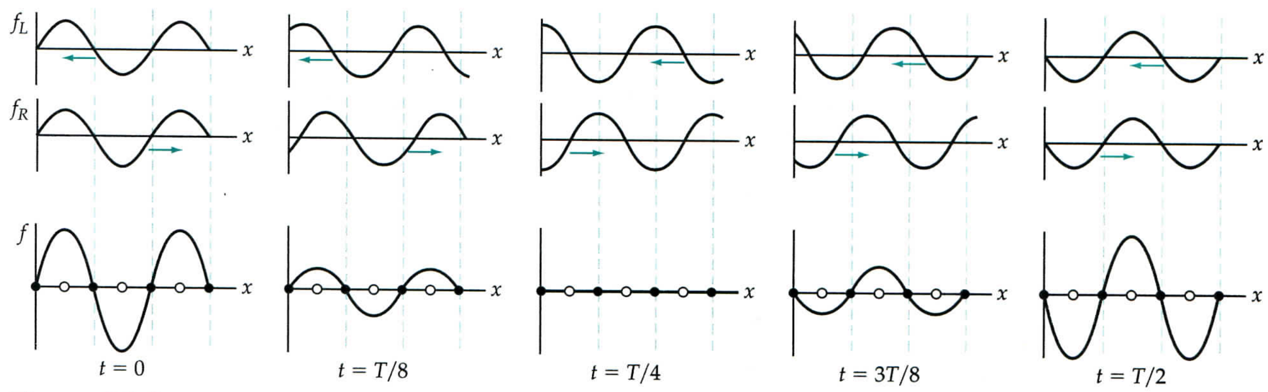



**Figure Q1.9**
How left-going and right-going sinusoidal waves combine to form a standing wave. Black dots correspond to nodes, white dots to antinodes.

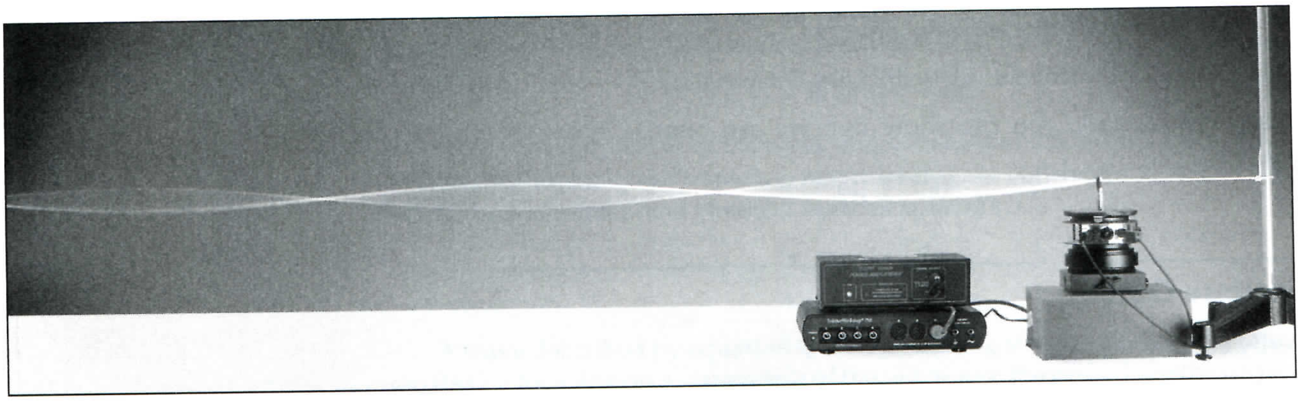


**Figure Q1.10**
A photograph of standing waves on a vibrating string.

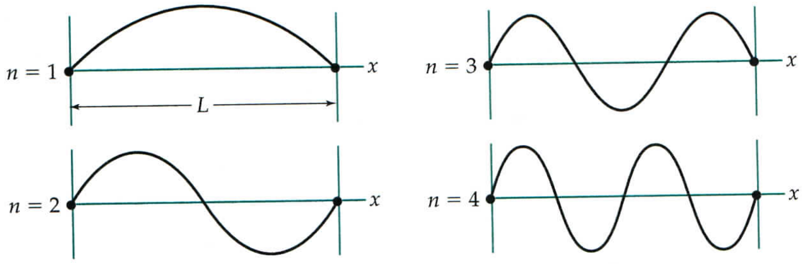



**Figure Q1.11**
Normal modes of oscillation of a string fixed at both ends. Note that $n$ corresponds to the number of half-wavelengths of the wave that fits between the fixed points.

where $v$ is the velocity of waves on the string. Note that the frequency $f$ of any viable standing wave on this string will be an integer multiple of the **fundamental frequency** $v/2L$, which happens to be the number of times a wave can move down the string and back (a distance of $2L$) per second.

We call the standing wave corresponding to each value of $n$ here a **normal mode** of the string's oscillation. The wave corresponding to $n = 1$ we call the string's **fundamental mode,** and its other modes are the **harmonics** of the fundamental mode.

## Exercise Q1X.6

Use equations Q1.2, Q1.4, and Q1.10 to fill in the missing steps leading to the last equality in equation Q1.11*a*.

If the left end of the string is fixed while the other end is *free,* we can still set up standing waves on this string, but the characteristics of the standing wave are somewhat different. Equations Q1.7 and Q1.9 still apply, but the condition that has to be satisfied at $x = L$ is not that $\sin kx = 0$ but rather that the *slope* of $\sin kx$ must be equal to zero. The slope of $\sin kx$ is zero at the maxima and minima of the sine function, which occur where $kx = \pi/2$, $3\pi/2$, $5\pi/2$, and so on. Therefore for a string with one fixed end and one free end (or for air in a tube with one open end and one closed end), we have

$$kL = n\frac{\pi}{2} \quad \Rightarrow \quad L = n\frac{\lambda}{4} \quad \text{and} \quad f = \frac{v}{4L}n \quad n \text{ is an odd integer} \tag{Q1.11b}$$

We see that the frequency of the normal-mode oscillations in this case is still an integer multiple of the fundamental frequency $v/4L$, but the fundamental frequency is *one-half* that of a string fixed at both ends, and only odd multiples of this frequency correspond to viable normal modes. Figure Q1.12 illustrates some of these modes.

So, in summary,

$$f = \frac{v}{2L}n \quad \text{where } n = 1, 2, 3, \ldots \quad \left(\begin{array}{c}\text{when the boundaries are analogous}\\ \text{to either two free or two fixed ends}\end{array}\right) \tag{Q1.12a}$$

$$f = \frac{v}{4L}n \quad \text{where } n = 1, 3, 5, \ldots \quad \left(\begin{array}{c}\text{when the boundaries are analogous}\\ \text{to one fixed and one free end}\end{array}\right) \tag{Q1.12b}$$

**Purpose:** These equations describe the frequencies $f$ of the fundamental modes of standing waves in a medium between two reflecting boundaries.

**Symbols:** $L$ is the distance between the boundaries, $v$ is the speed of traveling waves in the medium, and $n$ is a positive integer (a positive *odd* integer in the second case).

**Limitations:** This equation applies only when the medium is uniform and the waves are essentially one-dimensional. The boundaries must be perfectly reflecting if the wave is to sustain itself.

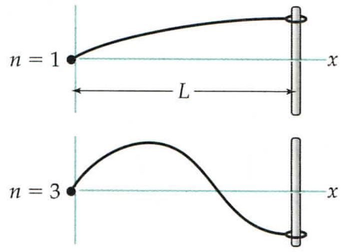



**Figure Q1.12**
The first two standing wave modes for a string with one end fixed and one end free. Note that in this case, an odd number of quarter-wavelengths must fit between the boundaries.

## Q1.6 The Fourier Theorem

In section Q1.5, we saw that traveling *sinusoidal* waves on a string can set up standing waves between the string's ends, *if* they have the right wavelength and frequency. What if the shape of the initial disturbance on the string is *not* sinusoidal?

Statement of the Fourier theorem

During the decade of the 1820s, Jean Baptiste Joseph Fourier showed that one can construct *any* periodic and reasonably continuous wave function $f(x, t)$ by superposing a sufficiently large number of sinusoidal waves with appropriately chosen amplitudes and frequencies. In particular, he showed that one can consider *any* waveform of a vibrating string to be a superposition of appropriately weighted sinusoidal waves corresponding to the *normal modes* of the string.

For example, imagine that we have a string fixed at both ends whose shape at time $t = 0$ is what one might call a **square wave**

$$f(x) = \begin{cases} +A & \text{if } 0 \le x < \frac{1}{2}L \\ -A & \text{if } \frac{1}{2} \le x \le L \end{cases} \tag{Q1.13}$$

It turns out that this wave is equivalent to the following infinite sum of sinusoidal waves (see figure Q1.13):

$$f(x) = A\frac{4}{\pi}\left(\sin kx + \frac{1}{3}\sin 3kx + \frac{1}{5}\sin 5kx + \cdots\right) \tag{Q1.14}$$

where $k = \pi/L$ is the wavenumber of the fundamental mode. Note that this sum involves only sinusoidal waves whose wavenumbers correspond to normal modes of a string fixed at both ends (see equation Q1.9). Each of these

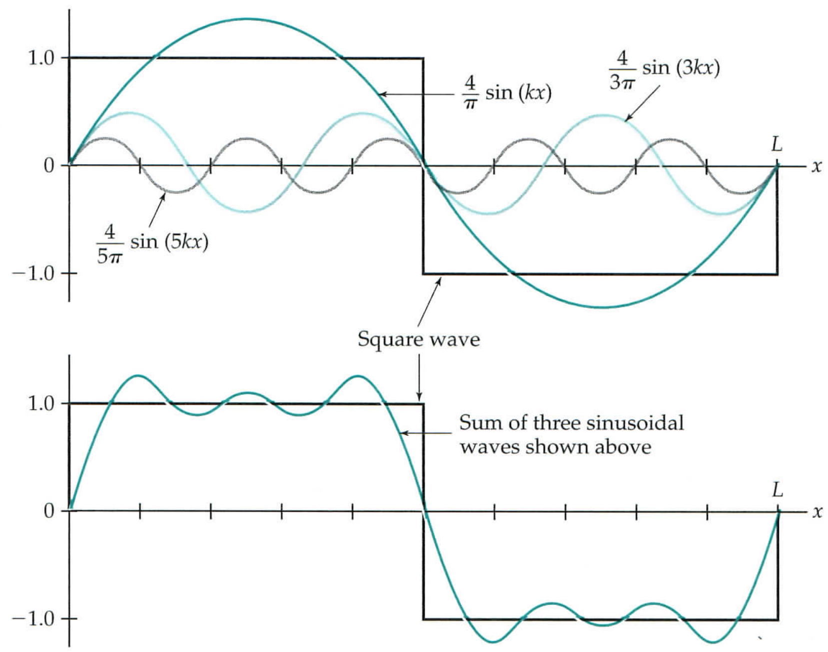



**Figure Q1.13**
This figure illustrates how we can construct a square wave from sinusoidal waves. The top graph shows the square wave and the sinusoidal waves corresponding to the first three terms of equation Q1.14. The bottom graph shows that the sum of just these three terms is a recognizable approximation to the square wave. The approximation gets better as more terms are added.

normal-mode standing waves, in turn, can be considered to be a sum of sinusoidal traveling waves (see equations Q1.6 and Q1.8).

Understanding sinusoidal waves is the key to understanding all waves

One implication of this theorem is that if we deeply understand the behavior of sinusoidal waves, we effectively understand the behavior of any kind of wave, since we can consider any wave to be a sum of sine waves. *This theorem is perhaps the single most important principle of wave behavior.* While its mathematical proof is beyond our means at present, you may well see it proved several times in different ways if you take higher-level physics or math classes (an indication of its importance). In our present context, this theorem implies that no matter what the shape of the initial disturbance of the string might be, we can think of its subsequent oscillation as being a combination of normal-mode oscillations.

## Q1.7 Resonance

The system responds most strongly to disturbances at normal-mode frequencies

Imagine that we disturb a string by wiggling it at a frequency different from one of its normal-mode frequencies. We will find that sometimes a given wiggle happens to be timed correctly to give the string some energy, but just as often the string is moving in such a way as to push back as we attempt to wiggle it, transferring energy back to us. The result is that on the average, our wiggling transfers very little energy to the string, even if we wiggle it violently. On the other hand, if we wiggle the string at one of its normal-mode frequencies, then we can time it so that when we tug on the string, the string is moving in just the right way to accept energy from the push. As a result, the string picks up more and more energy from our efforts, increasing its oscillation amplitude (until drag and other effects dissipate energy at the same rate as we supply it).

By way of analogy, imagine that you give a child on a swing a series of pushes at a rate of, say, two pushes per second. Pushes this frequent are not going to effectively get the child moving. Sometimes the child is moving forward as you push forward, and so you transfer energy to the child. Just as often, though, the child is moving backward, so your push actually *extracts* energy from the child's motion. Except for frustrating the child, your efforts do very little on the average.

On the other hand, if you synchronize your pushes with the natural motion of the swing (delivering them with a frequency of more like once every 2 s), you are always pushing on the swing when it moves forward, and therefore you always transfer energy *to* the swing instead of the reverse. Each successive oscillation, therefore, the swing will gain more energy from your push, and (to the child's delight) the amplitude of oscillation increases.

This tendency to react strongly to disturbances at normal-mode frequencies but ignore disturbances at other frequencies is a general feature of oscillating systems: we call this phenomenon **resonance.**

Musical instruments

The concepts of resonance and the Fourier theorem help us understand how many kinds of musical instruments work. For example, a bow sliding across a violin string disturbs the string in a complicated, random fashion. The Fourier theorem teaches us that we can think of this random disturbance as being a superposition of many *sinusoidal* disturbances having a wide range of frequencies and amplitudes. *Some* of these frequencies closely match the normal-mode frequencies of the violin string: the string thus extracts energy from the bow at these frequencies and begins to vibrate. The resulting wave on the string is a complex superposition of normal-mode oscillations at the string's fundamental frequency and integer multiples of that frequency. Our ears and brains process the complicated sound produced by the vibrating string as a musical tone at the fundamental frequency "colored" by the harmonics, which give it the distinctive sound that makes us think "violin."

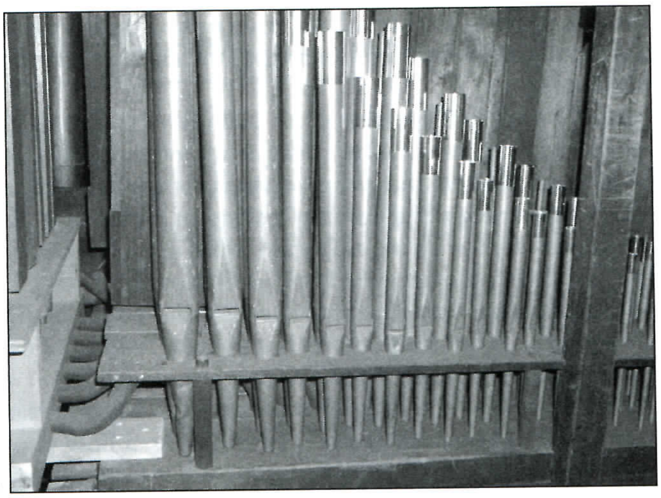
Many musical instruments use columns of air in tubes instead of strings as the resonating system that creates the sound. When you blow across the top of a bottle, for example, you are randomly disturbing the air trapped in the bottle, and some of the energy in the hiss of the air goes to exciting normal modes of the air in the bottle, creating a tone.

Organ pipes work in a similar manner: a stream of air blowing against a sharp edge creates vortices that disturb the air in the pipe in a complex way. An organ pipe that is open at both ends (see figure Q1.14a) is analogous to a string fixed at both ends, and so the sound generated by the pipe consists of a tone at the fundamental frequency of the air in the pipe plus contributions from all higher harmonics. This gives the pipe a rich sound that is much like that of a violin.

Other kinds of organ pipes are closed at one end (see figure Q1.14b), which is analogous to a string that is free at one end (see figure Q1.8). Not only does this give the pipe a fundamental frequency that is *one-half* that of a doubly open pipe of equivalent length, but also the air in a closed pipe can only vibrate at the odd harmonics of its fundamental tone (see equation Q1.12*b*). This absence of even harmonics gives the sound produced by such pipes a very distinctive "flutey" tone color.

Resonances during earthquakes

Finally, the phenomenon of resonance helps explain why light, small wooden buildings generally do very well in an earthquake, while nearby buildings that are six stories tall (and certain multiples of six stories) can be severely damaged. A six-story building constructed using ordinary techniques happens to have a fundamental natural frequency of oscillation that is close to that of a certain type of earthquake wave, so these buildings effectively absorb energy and oscillate wildly in response to an earthquake. The same waves can effectively transfer energy to higher-frequency normal modes in a building whose height is an appropriate multiple of six stories. Small, wooden-frame buildings, on the other hand, have a much higher


(a)

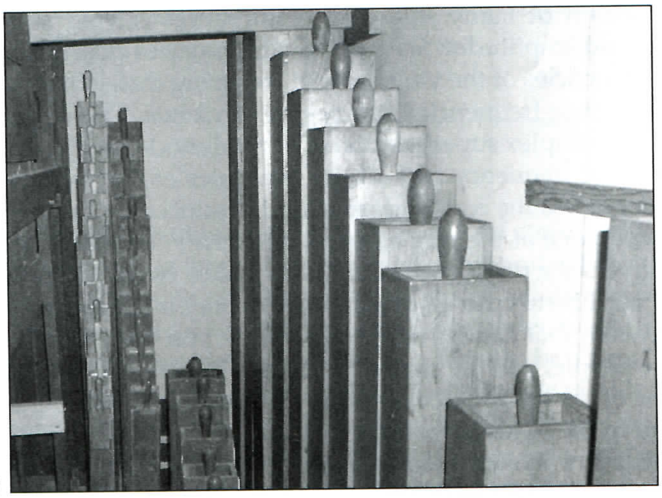

(b)

**Figure Q1.14**
(a) A photograph showing organ pipes that are open at both ends. (b) A photograph showing organ pipes that are closed at one end. (The handle at the end of each allows a tuner to adjust the length of the air column in the pipe.)

fundamental frequency of oscillation and so do not effectively absorb energy from an earthquake.

We will see in chapters Q7 and Q8 that resonance helps us understand the peculiar behavior of bound quantum systems.

## Example Q1.1

*Problem* The air column in a certain organ pipe open at both ends has a fundamental frequency of 260 Hz (this corresponds to middle C). How long must such a pipe be?

*Model* The air column in an organ pipe open at both ends is analogous to a string fixed at both ends. According to equation Q1.10, the fundamental frequency of such a system is given by

$$f = \frac{v}{2L} \tag{Q1.15}$$

where in this case $L$ is the length of the air column (which is the length of the pipe) and $v$ is the speed of a wave moving in this column (which is the speed of sound, or 343 m/s at normal temperature and pressure).

*Solution* Therefore, the required length of the pipe is

$$L = \frac{v}{2f} = 0.66 \text{ m} \approx 2.2 \text{ ft} \tag{Q1.16}$$

(The pipe will actually be a bit shorter than this: the effective "fixed end" of the air column is actually a bit outside of the open end.)

## TWO-MINUTE PROBLEMS

**Q1T.1** A linear traveling wave can be partially reflected when it encounters another linear traveling wave, true (T) or false (F)?

**Q1T.2** A sound wave traveling in air hits the surface of a body of water. Is the reflected wave (A) inverted or (B) upright? (Make a guess). The reflection will be total, T or F?

**Q1T.3** Imagine that we create a traveling compression wave in a spring that has one end fixed. When the wave reflects from the fixed end, it will be inverted, T or F?

**Q1T.4** Imagine that you are near one end of a 150-m-long cylindrical tunnel open to the air on both ends. If you give a shout, you might hear an echo, T or F?

**Q1T.5** If you face a cliff or a large concrete wall and give a shout, you will hear an echo. Are the sound waves of the echo inverted or upright compared to the waves of your original shout?
A. Inverted
B. Upright

**Q1T.6** The frequencies of the normal modes of a string that is free at both ends are the same as those of a string that is fixed at both ends, T or F?

**Q1T.7** A sinusoidal standing sound wave inside a tube that is open at both ends must fit between the tube's ends
A. An integer number of wavelengths
B. An integer number of half-wavelengths
C. An odd integer number of quarter-wavelengths

**Q1T.8** When the frequency of a standing wave on a string is doubled, its wavelength is multiplied by a factor of
A. $\frac{1}{4}$
B. $\frac{1}{2}$
C. $1/\sqrt{2}$
D. $\sqrt{2}$
E. 2
F. 4
T. $\lambda$ is unchanged

**Q1T.9** The period $T$ of the fundamental mode of the air in a pipe open at one end and closed at the other is equal to what multiple or fraction of the time $\Delta t$ required for a sound wave to travel from one end of the tube to the other?
A. $T = \frac{1}{4}\Delta t$
B. $T = \frac{1}{2}\Delta t$
C. $T = \Delta t$
D. $T = 2\,\Delta t$
E. $T = 4\,\Delta t$
F. Other (specify)

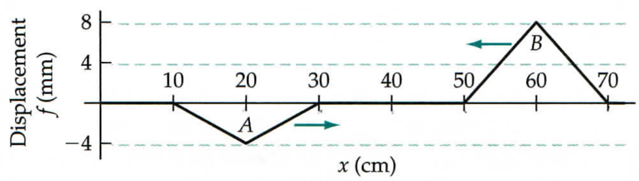
**Q1T.10** A certain organ pipe is open at both ends. Another pipe of the same length is open at one end and closed at the other. Which will have the lower pitch?
A. The pipe open at both ends.
B. The pipe closed at one end.
C. Both will have the same pitch.
D. It depends on the pipes' diameters.

**Q1T.11** A certain stretched string has a fundamental frequency of 165 Hz (E below middle C). If someone sings a note at a frequency of 495 Hz (B above middle C), the string will *not* respond significantly to the disturbing influence of the sound waves, T or F?

## HOMEWORK PROBLEMS

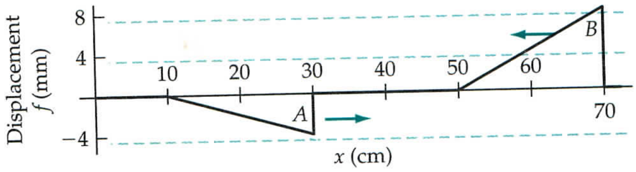
### Basic Skills

**Q1B.1** Consider the triangle-shaped waves shown in the drawing below. Each wave moves with a speed of 5 cm/s in the direction indicated. Draw separate graphs showing what the superposition principle



implies that the combined wave should look like at $t = 2$ s, 3 s, 4 s, and 6 s.

**Q1B.2** Consider the sawtooth-shaped waves shown in the drawing below. Each wave moves with a speed of 5 cm/s in the direction indicated. Draw separate graphs showing what the superposition principle implies that the combined wave should look like at $t = 2$ s, 3 s, 4 s, and 6 s.




**Q1B.3** Imagine that we have a string 1.2 m long and fixed at both ends. We adjust the tension on the string until the speed of waves on the string is 24 m/s. What is the frequency of the string's fundamental mode of oscillation?

**Q1B.4** An organ pipe open at both ends is 2.2 m long. What is the frequency of the fundamental mode of the air in the pipe?

**Q1B.5** An organ pipe open at both ends has a fundamental frequency of 440 Hz (concert A). What is the length of this pipe? What are the frequencies of its first three harmonics?

**Q1B.6** An organ pipe closed at one end has a fundamental frequency of 220 Hz (A below middle C). What is the length of this pipe? What are the frequencies of its first three harmonics?

**Q1B.7** Imagine that we have a string 1.5 m long that is fixed at both ends. We adjust the string's tension so that the string's fundamental frequency is 100 Hz. What is the frequency of the normal mode of the string's oscillation that has three antinodes?

**Q1B.8** Imagine that we have an organ pipe closed at one end. The length of the pipe is such that the fundamental frequency of the air in the pipe is 230 Hz. What is the frequency of the normal mode of the air having two internal antinodes (not counting the antinode at the closed end)?

## Synthetic

**Q1S.1** When you tune a woodwind instrument, you pull apart or push together two sections of the instrument.
(a) Why does this change the instrument's pitch?
(b) How is this related to the purpose of the slide on a trombone?

**Q1S.2** Imagine that a string on an acoustic guitar is 25 in. long between its fixed ends. According to example E15.2, the speed of waves on a stretched string is $v = \sqrt{F_T/\mu}$. The highest E string on such a guitar has a pitch of about 329 Hz. Assume that the particular string used has a mass per unit length of $\mu = 0.2$ g/m.
(a) What tension force must be applied to this string?
(b) By what fraction would we have to increase the tension to tune the string up to G (392 Hz)?

**Q1S.3** Here is a way to demonstrate the Fourier theorem. Find a piano and open it so that you can clearly hear the strings. Press the damper (often the rightmost) pedal: this allows the strings to vibrate freely. Now sing "uuuuu" loudly but at a definite pitch for a brief time. You should be able to hear the strings play the same note back to you. If you touch various strings with your finger, you may be able to convince yourself that only the one set of two or three strings is significantly vibrating, the set closest in natural frequency to the pitch you sang. The other strings essentially did not respond to your note, since they did not have the right natural frequency to resonate with it.

Now clap your hands loudly, or slam a book on the floor, or otherwise make a sudden loud sound with no definite pitch. What do the piano strings do? How does what you hear imply that we can think of the single complicated pulse of your clap (or whatever), even though it has no discernible pitch, as the sum of sine waves that *do* have definite frequencies?

**Q1S.4** A concert flute (see figure Q1.15) is about 2 ft long, and its lowest pitch is middle C (about 261 Hz). Should we consider a flute to be a pipe that is open at both ends or at just one end? (One end of the flute seems to be clearly closed, so if you choose the former, you should try to explain where the other open end is.)

**Figure Q1.15**
A photograph of a standard concert flute. Does this represent a tube open at one end or both ends? (See problem Q1S.4.)

**Q1S.5** You may know that if you inhale helium, your voice sounds strange and high-pitched if you talk as you exhale the helium. Why is this? (*Hints:* Your sinuses are resonating chambers of air that emphasize certain of the pitches produced by your vocal cords. The speed of sound is almost 3 times higher in helium than in normal air.) *Note:* Inhaling helium can be dangerous because while the helium is in your lungs, your body is not getting the oxygen it needs to survive.

**Q1S.6** Imagine that you have a string that has one end that is fixed and the other end is perfectly absorbing, so that traveling waves moving past that end are not reflected at all. Imagine that we wiggle the string sinusoidally near the absorbing end (this will send traveling sinusoidal waves in both directions along the string). Can we set up standing waves on this string? If so, at what frequencies (or is there *any* limit on the frequency)?

**Q1S.7** Consider an organ pipe 34.3 cm long that has one end open and one end closed. What is the fundamental pitch of this pipe? Where are the nodes (relative to the closed end) of the normal mode of the air in this pipe whose frequency is 1250 Hz?

**Q1S.8** Consider an organ pipe 1.72 m long that has one end open and one end closed. What is the fundamental pitch of this pipe? Where are the nodes (relative to the closed end) of the normal mode of the air in this pipe whose frequency is 150 Hz?

**Q1S.9** In the equal-temperament tuning system (the most common system today for tuning musical instruments), each half-step on the musical scale has a frequency that is $2^{1/12}$ higher than the previous note.

(a) Make a list of the frequencies of the 12 half-steps (C#, D, D#, E, F, F#, G, G#, A, A#, B, C) above middle C, given that A is defined to be 440 Hz.
(b) Argue that any note 12 half-steps above another note will have exactly twice the frequency of the lower note. (We describe such pitches as being an *octave* apart.)
(c) Certain combinations of notes sound "harmonic" because their frequencies are very nearly simple integer ratios of each other. As an example, consider a C major chord (C, E, G). What are the simplest ratios that are close to the actual ratios of the frequencies of E to C and G to C? (In the equal-temperament tuning system, these ratios are not quite exact. Other tuning systems make these ratios more pure in certain keys, but the equal-temperament system, because of its symmetry, has the advantage that no key is favored.)
(d) Sets of notes with simple frequency ratios also correspond to the harmonic frequencies of a single note. If C, E, and G are adjacent harmonics of some fundamental tone, what is the frequency of that tone?

**Q1S.10** The speed of the wave on a flexible string can, if you think about it, depend on only two quantities: the tension force $F_T$ on the string (which tells you how strongly the string is pulled back toward equilibrium when it is disturbed) and the mass per unit length $\mu$ of the string (which tells how quickly or slowly the string *responds* to that restoring push). Let us guess that the speed depends on some power of $F_T$ multiplied by some power of $\mu$. Show that if this is so, the speed $v$ of a wave on the string *must* depend on these quantities as follows

$$v = C\sqrt{\frac{F_T}{\mu}} \tag{Q1.17}$$

(where $C$ is an unknown constant with no units), since this is the only such way to combine these quantities that yields the correct units. (In chapter E15, we did a much more difficult formal derivation of this wave speed and found that $C = 1$.)

**Q1S.11** The speed of a sound wave in air plausibly depends on the ambient pressure $p_0$ of the air (which expresses how strongly a bit of compressed air wants to return to equilibrium) and the ambient density $\rho_0$ of the air (which expresses how slowly or quickly the air responds to pressure changes). Assuming that the sound velocity is a product of powers of these quantities, find the only possible such product that has the correct units. Since standard air pressure is $1.0 \times 10^5$ N/m$^2$, the density of air at this pressure and 20°C is 1.2 kg/m$^3$, and the speed of sound has a measured value of 343 m/s under such circumstances, determine the value of any unitless constant that might be in your equation. (*Hint:* Use dimensional analysis.)

## Rich-Context

**Q1R.1** You and a companion are trying to escape from some bad guys one dark night. With the sounds of pursuit close behind, you come upon an open pipe that appears to cut through a hillside. If the pipe is open at the other end, you may be able to escape this way. If it is closed, you will be trapped. Your companion says, "I know how to find out," and sticks his head in, yells something, and then listens. He then pulls his head out and says, "It must be open at the other end. When I yelled 'hello,' I heard the reflection come back 'olleh.' Thus the reflection was inverted, and since the open end of a pipe is like the fixed end of a string, it will reflect the sound inverted." With a thrill of fear, you realize that your companion is *lying* to you and thus is possibly in cahoots with the bad guys. How do you know this?

**Q1R.2** A bugle (see figure Q1.16) is simply a coiled length of pipe, without slides or valves. One plays different notes on a bugle by buzzing one's lips at different frequencies.

**Figure Q1.16**
A photograph of a bugle (see problem Q1R.2).

(a) How does this change the pitch (frequency) of the sound the bugle makes?
(b) The bugle can play only certain pitches and not others (for example, think of the piece "Taps," which is entirely constructed of only four different pitches). What are these allowed pitches, and why are other pitches impossible?
(c) Imagine that certain bugle plays in C, so that the pitches in "Taps" are low G (196 Hz), middle C (261 Hz), E (329 Hz), and G (392 Hz). How long would this bugle be if uncoiled? (*Hint:* Is the bugle effectively a tube with two open ends or a tube with one open end and one closed end? Think about the implications of either model. How could you get the pitches listed if the bugle is open at one end and closed at the other?)

## Advanced

**Q1A.1** Here is yet another way to derive equation Q1.17 in problem Q1S.10: Consider a pulse traveling left down a string at a constant velocity $v$. Imagine that we look at this situation from a reference frame where the pulse is at rest, and the string is moving to the right with speed $v$. This frame will be inertial, so we can apply Newton's laws. Imagine that we focus our attention on a little bit of string of length $dL$ passing the crest of the pulse. We can find a circle with some radius $R$ that approximates the curvature of this bit of string (see figure Q1.17). Argue that the net downward force on this piece of string is roughly $F_{net} = F_T(dL/R)$ (ignoring gravity). Since this force is what constrains the bit of string to follow a circular path of radius $R$ as it moves over the crest with speed $v$, this net force must be equal to $mv^2/R$ by Newton's second law, where $m$ is the mass of the bit of string. Show that combining these equations implies equation Q1.17, with $C = 1$. (*Hint:* Note that if $dL$ is small, then $\theta$ is small, implying that $\sin\theta \approx \theta$.)

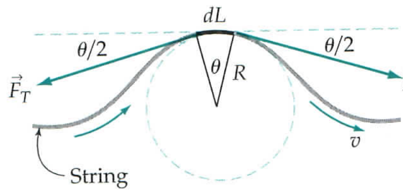
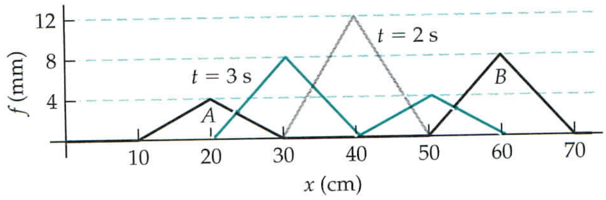



**Figure Q1.17**
A drawing illustrating how we can treat a small part of a string mass undergoing circular motion when the crest of the pulse wave passes.

## ANSWERS TO EXERCISES

**Q1X.1** The total wave function looks as shown below at time $t = 2$ s (gray lines) and $t = 3$ s (colored lines):




**Q1X.2** The edge of the tub will be analogous to a free end of a string or the closed end of a tube: the water's amplitude is not limited, but the wave cannot continue beyond the boundary. Therefore, the reflected wave will be upright.

**Q1X.3** Consider again the crest that corresponds to the argument of the sine function being $\pi/2$:

$$kx_{crest} + \omega t = \frac{\pi}{2} \quad \Rightarrow \quad x_{crest} = \frac{\pi}{2k} - \frac{\omega}{k}t \tag{Q1.18}$$

Taking the time derivative of both sides of this equation, we find that $v_x \equiv dx_{crest}/dt = -\omega/k$.

**Q1X.4** Since $\omega = 2\pi/T$ and $k = 2\pi/L$, we have

$$v = |v_x| = +\frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T} \tag{Q1.19}$$

Since $1/T = f$, this also implies that $v = \lambda f$.