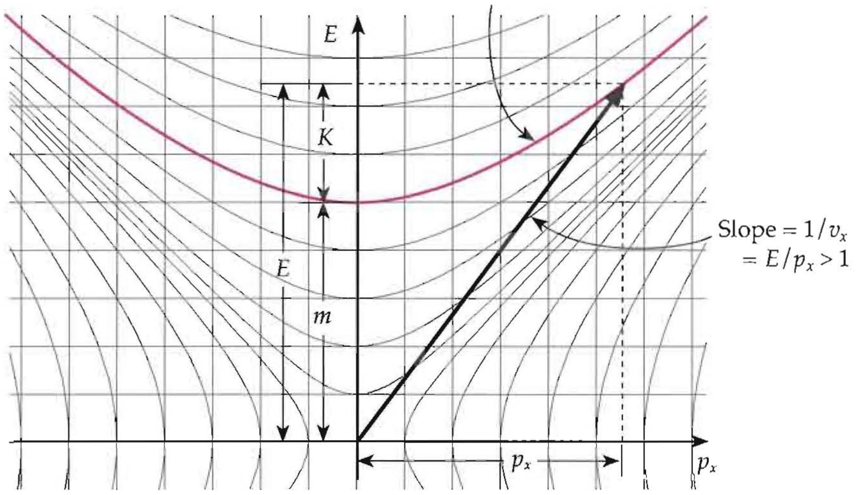
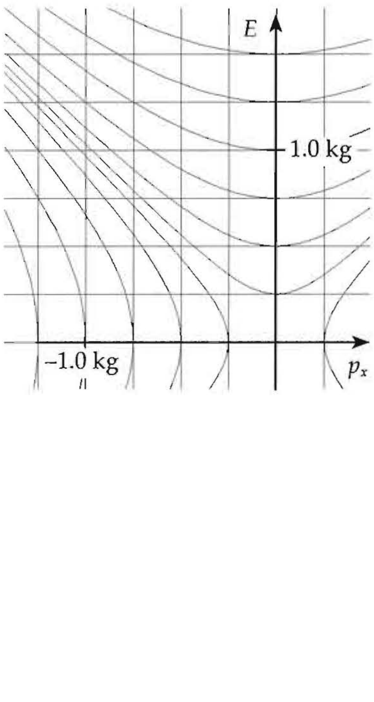


Figure R9.5
Virtually everything that you need to know about four-momentum diagrams. No matter what the x-velocity of an object of mass m might be, the tip of the arrow representing its four-momentum lies on the hyperbola $E^2 - |\vec{p}|^2 = m^2$. The inverse slope of the arrow representing the four-momentum is equal to v_x, which always has a magnitude less than 1.



Exercise R9X.1

On the hyperbola graph paper to the left, draw an energy-momentum diagram of an object whose mass is 1.0 kg and which moves in the $-x$ direction at a speed of $\frac{3}{5}$. Read its relativistic energy E, its relativistic kinetic energy K and the x component of its four-momentum p_x from the diagram.

R9.2 Solving Conservation Problems

Solving conservation problems algebraically

The law of conservation of four-momentum (like the law of conservation of ordinary momentum) is most useful when applied to an isolated system of objects undergoing some type of *collision* process (i.e., a kind of sudden interaction between the objects in the system that may be strong and complicated but limited in time). In such a case, the system has a clearly defined state "before" and "after" the collision, making it easy to compute the system's total four-momentum both before and after the collision. The law of conservation of four-momentum states that the system should have the *same* total four-momentum after the collision process as it had before.

What does this really mean mathematically? Since four-momentum is a (four-dimensional) vector quantity, conservation of four-momentum means that *each component of the system's total four-momentum is separately conserved.* For example, consider a system consisting of two objects, and let the objects' four-momenta before the collision be $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and after the collision be $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$. Conservation of four-momentum then requires that

$$\begin{bmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{bmatrix} + \begin{bmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{bmatrix} + \begin{bmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{bmatrix} \tag{R9.3}$$

remembering that the time component of a four-momentum vector (i.e., the relativistic energy) is usually given the more evocative symbol E instead of p_t. As usual, each row of this equation must be *separately* true for four-momentum to be conserved.

Example R9.1

Problem: Suppose that somewhere in deep space a rock with mass $m_1 = 12$ kg is moving in the $+x$ direction with $v_{1x} = +\frac{4}{5}$ in some inertial frame. This rock then strikes another rock of mass $m_2 = 28$ kg at rest ($v_{2x} = 0$). Pretend that the first rock, instead of instantly vaporizing into a cloud of gas (as any *real* rocks colliding at this speed would), simply bounces off the more massive rock and is subsequently observed to have an x-velocity $v_{3x} = -\frac{5}{13}$. What is the laager rock's x-velocity v_{4x} after the collision?

Solution The first step in solving this problem is to calculate the energy E_1 and the x-momentum p_{1x} of the smaller rock before the collision. Using the definitions of these four-momentum components, we find that

$$E_1 \equiv \frac{m_1}{\sqrt{1-v_{1x}^2}} = \frac{m_1}{\sqrt{1-\frac{16}{25}}} = \frac{m_1}{\sqrt{\frac{9}{25}}} = \frac{5}{3}(12\text{ kg}) = 20\text{ kg} \tag{R9.4a}$$

$$p_{1x} \equiv \frac{m_1 v_{1x}}{\sqrt{1-v_{1x}^2}} = \frac{m_1(+\frac{4}{5})}{\frac{3}{5}} = +\frac{4}{3}(12\text{ kg}) = +16\text{ kg} \tag{R9.4b}$$

Similarly, the larger rock's energy and x-momentum before the collision are

$$E_2 \equiv \frac{m_2}{\sqrt{1-v_{2x}^2}} = \frac{m_2}{\sqrt{1-0^2}} = m_2 = 28\text{ kg} \tag{R9.5a}$$

$$p_{2x} \equiv \frac{m_2 v_{2x}}{\sqrt{1-v_{2x}^2}} = \frac{m_2(0)}{\sqrt{1-0^2}} = 0\text{ kg} \tag{R9.5b}$$

The smaller rock's energy and x-momentum *after* the collision are

$$E_3 \equiv \frac{m_1}{\sqrt{1-v_{3x}^2}} = \frac{m_1}{\sqrt{1-(-\frac{5}{13})^2}} = \frac{m_1}{\sqrt{\frac{144}{169}}} = \frac{13}{12}(12\text{ kg}) = 13\text{ kg} \tag{R9.6a}$$

$$p_{3x} \equiv \frac{m_1 v_{3x}}{\sqrt{1-v_{3x}^2}} = \frac{m_1(-\frac{5}{13})}{\frac{12}{13}} = -\frac{5}{12}(12\text{ kg}) = -5\text{ kg} \tag{R9.6b}$$

Conservation of four-momentum requires that the four-momentum vectors before the collision add up to the same value after the collision, meaning that $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4$ or

$$\boldsymbol{p}_4 = \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 = \begin{bmatrix} 20\text{ kg} \\ 16\text{ kg} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 28\text{ kg} \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 13\text{ kg} \\ -5\text{ kg} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 35\text{ kg} \\ 21\text{ kg} \\ 0 \\ 0 \end{bmatrix} \tag{R9.7}$$

Knowing the energy and x-momentum of an object is sufficient to determine both its energy and x-velocity. Using equation R8.31, we see that the larger rock's mass is still

$$m = \sqrt{E_4^2 - p_{4x}^2} = \sqrt{(35\text{ kg})^2 - (21\text{ kg})^2} = (7\text{ kg})\sqrt{5^2 - 3^2} = 28\text{ kg} \tag{R9.8a}$$

after the collision. (Since energy is not being transformed to other forms, this is an elastic collision.) According to equation R8.33, its final x-velocity is

$$v_{4x} = \frac{p_{4x}}{E_4} = \frac{+21\text{ kg}}{35\text{ kg}} = +\frac{3}{5} \tag{R9.8b}$$

This completes the solution.

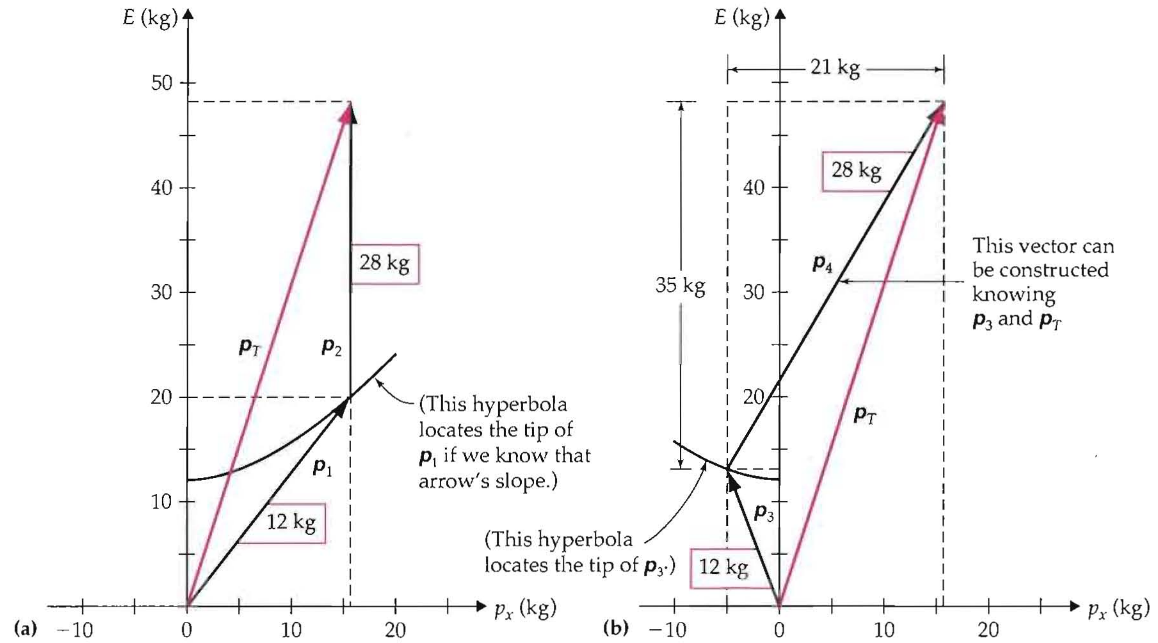


Figure R9.6
(a) The four-momenta of the rocks *before* the collision. The vector sum of these four-momenta is represented by the arrow $\boldsymbol{p}_T$. Since the four-magnitudes of the individual four-momentum arrows (which equal the masses of the corresponding rocks) cannot be read directly from the diagram, I have adopted the expedient of attaching a "flag" to each four-momentum arrow that states its magnitude. (b) The four-momenta of the rocks *after* the collision. The vector sum of these four-momenta is still $\boldsymbol{p}_T$ by four-momentum conservation. Since we know $\boldsymbol{p}_3$, we can construct the unknown four-momentum arrow for $\boldsymbol{p}_4$, read its components from the diagram as shown, and use these to compute the corresponding rock's mass and x-velocity.

Example R9.2:

Problem: Solve the rock collision problem discussed in example R9.1, using an energy-momentum diagram. (Remember that a rock with mass $m_1 = 12$ kg moving with $v_{1x} = \frac{3}{5}$ hit a rock with mass 28 kg at rest. After the collision, the first rock, whose mass is unchanged, moves with x-velocity $v_{3x} = -\frac{5}{13}$.)

Solution Since the sum of four-momentum arrows is defined as the sum of ordinary vectors (we simply add the components), we can add four-momenta arrows on an energy-momentum diagram just as we would ordinary vector arrows (by putting the tail of one vector on the tip of the other while preserving their directions). Using this technique, we see in figure R9.6a that in the rock example, the system's total four-momentum *before* the collision has components $E_T = 48$ kg and $P_{Tx} = 16$ kg. The two rocks' four-momentum arrows after the collision have to add up to the *same* total four-momentum arrow; and since we know the smaller rock's four-momentum after the collision, we can *construct* the larger rock's final four-momentum arrow $\boldsymbol{p}_4$ (figure R9.6b). We can then read the components of this arrow right off the diagram, getting the same results as in equations R9.7. One can then use these results (as we did before) to compute that rock's mass and x-velocity.

Advantages and disadvantages of the graphical method

This graphical approach to the problem can be somewhat easier than the algebraic approach if one has access to hyperbola graph paper (which